Heinrich von Stackelberg on joint production

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1. Introduction

Heinrich von Stackelberg’s (1905–46) exhaustive analysis of joint production in his *Grundlagen einer reinen Kostenlehre* (‘Foundations of a Pure Theory of Costs’, 1932) attracted wide attention immediately after it appeared. In fact, von Stackelberg’s *Kostenlehre* was recognised as an authoritative statement of cost theory both among German speaking economists and within the international economic community (Eucken 1948: 133, Hicks 1935, Schneider 1933, Tinbergen 1933, Niehans 1992: 94). Möller (1949: 401) even judged that ‘Stackelberg has the merit of having formulated the theory [of costs under single and joint production] in a definite, complete, elegant and exact . . . way’.

Notwithstanding the great interest in von Stackelberg’s *Kostenlehre* among contemporary economists, his contribution to the theory of joint production fell into oblivion later on. With the establishment of modern general equilibrium theory in the 1950s, and the concept of the so-called production set, joint production as an explicit issue did not play a major role within the mainstream of economic theory (Baumgärtner 2000: 130ff.). Only in the 1970s did the insight that ‘[t]he world of single-product firms with U-shaped average cost curves simply is not the world of reality’ (Baumol *et al.* 1988: vi) lead to a renewed interest in the explicit study of multi-output firms and industries. The issue of multi-output firms and industries has now found its way into many textbooks on industrial organisation (e.g. Tirole 1988).

The renewed interest in joint production within the literature on industrial organisation half a century after von Stackelberg’s contribution actually came as a rediscovery of this issue. Yet, none of the modern contributions,
such as e.g. Bailey and Friedlaender (1982), Baumol et al. (1988) or Tirole (1988), makes any reference to von Stackelberg’s work on joint production, maybe because it has only been available in German. However, the modern theory of multi-output firms employs concepts that were pioneered by von Stackelberg. While von Stackelberg today ranks as one of the most important German speaking economists of the twentieth century, who initiated the reorientation of German economics to the Anglo-Saxon mainstream (Krelle 1987: 470, Niehans 1992), the importance of his contribution to the theory of joint production is still widely neglected.\(^5\)

The aim of this paper is threefold: (i) to outline Heinrich von Stackelberg’s contribution to the theory of costs under joint production given in his *Kostentheorie* and, thereby, to draw the attention of the English speaking readership to this work; (ii) to assess critically the place of his contribution in the modern history of the theory of joint production; and (iii) to suggest an answer to the question of why von Stackelberg’s theory of joint production, although immediately raising a lot of attention, eventually fell into oblivion and, to some extent, contributed to the abandonment of the issue within the mainstream of economic theory for decades.\(^6\)

The paper is organised as follows. In Section 2, von Stackelberg’s contribution to the theory of joint production is placed in an historical context. Section 3 presents an overview of the treatment that von Stackelberg gives to the issue of joint production in his *Grundlagen einer reinen Kostentheorie* (1932). Von Stackelberg’s own programmatic (mis)interpretation of his results is critically reconsidered in Section 4. Section 5 assesses von Stackelberg’s contribution to the theory of costs under joint production, by comparing it to modern contributions to firm behaviour and industry structure under joint production. Section 6 concludes.

2. **Historical context of von Stackelberg’s contribution**

The analysis of joint production has long played an important role in the development of economic theory. Adam Smith, John Stuart Mill, Karl Marx, Johann Heinrich von Thünen, William Stanley Jevons, Alfred Marshall – all devoted a considerable amount of effort to the analysis of joint production. As a matter of history, the analysis of joint production ‘played a significant role in the gradual abandonment of the classical approach to the theory of value and distribution’ (Kurz 1986: 2). Classical value theory was pervaded by the distinction between two kinds of goods (Niehans 1990: 241): goods with a fixed supply, such as land, and goods with a variable supply, such as cloth. In the first case, quantity is fixed and the price depends only on demand. In the second case, the price is determined solely
by costs of production and demand only affects the quantity produced. For Adam Smith joint products, such as wool and mutton, did not fit into this scheme. He recognised the explanation of value of joint products as being special and, consequently, analysed them as a ‘third sort of produce’ (Smith 1976[1776]: 254–5). From then on, joint products were viewed from the analytical point of view – irrespective of their empirical relevance – as a ‘peculiar case’, as John Stuart Mill (1965[1848]: 569) called them.

Alfred Marshall’s Principles of Economics (1925[1890]) established what is today called the neoclassical explanation of value. Marginalism appeared to be a general principle for the determination of equilibrium. At its centre is the conviction that substitution possibilities do, indeed, exist in both consumption and production. Marshall (1925[1890]: 390) argued that this also holds for the case of joint production (with flexible proportions of the outputs) and that, consequently, one might define the marginal cost of production separately for each of the joint products. Marshall simultaneously determined the equilibrium prices and quantities of both joint products given the demand curves for both of them and the supply curve for the entity from which both joint products emanate. That is, Marshall performed a partial equilibrium analysis of joint production, while the classical writers (Smith, Mill) had always studied the problem within a total analysis. Marshall’s conclusion was that joint production need no longer be held to be a peculiar case which deserves special methods for the explanation of value. Instead, the explanation of the value of joint products follows the same general explanatory principle that holds for all goods whatsoever:

[T]he general principle can be applied, that the relative proportions of the joint products of a business should be so modified that the marginal expenses of production of either product should be equal to its marginal demand price.

(Marshall 1925[1890]: 395)

While Marshall had formally analysed the simultaneous determination of prices and quantities of joint products in a partial equilibrium framework, the focus of attention then shifted to the decision problem of a multi-output firm that faces given prices for its outputs (Marshall 1970[1911], Edgeworth 1925a[1911]: 84–91, Fisher 1892). Typically, the firm was conceptualised by its cost function $K(x_1, x_2)$, which depends on output quantities $x_1$ and $x_2$. One example receiving a lot of attention was railway transportation of passengers and cargo freight (Pigou 1912: Part II, Ch. xiii, §3; Edgeworth 1925b[1915]: 485–91). The discussion of this example mainly served to clarify the distinction between different cases of joint production. For instance, Edgeworth (1925a[1911]: 86–7; similarly 1925b[1915]: 487) distinguished between ‘rival’, independent and ‘joint’ production by the criterion that $\partial^2 K/\partial x_1 \partial x_2 > (=, <) 0$. But apart from clarifying terminology the analysis did
not improve much on the insights already obtained by Marshall. The contribution of Irving Fisher (1892) was innovative insofar as he introduced graphical instruments, i.e. isocost curves and isorevenue curves, for the analysis of the decision problem of a multi-output firm.

Marco Fanno (1999[1914]), in his formal analysis of joint production, made a first attempt to go beyond the limited scope of Marshallian partial equilibrium analysis. He fully recognised the merits of general equilibrium theory and studied an approximated version of a general equilibrium system. Within this approximation, he was concerned with the supply at joint cost in the cases of perfect competition and monopoly, on both closed and open markets.

This was the state of research when Heinrich von Stackelberg started to work on production and price theory. Von Stackelberg has been described as ‘the most gifted theoretical economist in Germany during his time’ (Krelle 1987: 469). The attention of contemporary economists was, in particular, caught by the formal rigour and exactness of his treatment and the elegance and virtuosity with which he applied theoretical methods to economic problems (Schneider 1933, Tinbergen 1933). His textbook *Grundzüge der theoretischen Volkswirtschaftslehre* (1943) has been ranked as the first ‘modern’ introduction to economics (Krelle 1987: 469), in the sense that it is based on a coherent theory of household and firm behaviour.

Although the mathematical analysis of economic problems was a characteristic property of almost all his publications, mathematics for von Stackelberg was never an end in itself (Möller 1949: 399). Von Stackelberg recognised the danger that a mathematical treatment might lead to over-simplified theories or irrelevant abstractions and, consequently, ‘became more and more anxious for closer contact with the facts of real life’ (Eucken 1948: 133). The striving for realistic descriptions also shows in his contributions to production theory and price theory. Unlike many other economists at the time, von Stackelberg did not confine his analysis to the case of single production, which is straightforward to grasp within a mathematical formalism, but took the fact of joint production seriously. An example is his proof (von Stackelberg 1938a) that the factor demand function of a profit-maximising firm is always monotonically decreasing in the factor price, and that the firm’s supply function is always monotonically increasing in the output price. Von Stackelberg (1938a: 83), in his assumptions for this proof, explicitly allows for joint production with fixed and with flexible output proportions.

### 3. A theory of costs under joint production

Heinrich von Stackelberg presents a formal and rigorous treatment of joint production in his *Grundlagen einer reinen Kostenlehre* (‘Foundations of a
Pure Theory of Costs’, 1932). The aim of that book is to study the role that costs of production play in determining the optimal production programme of a profit-maximising firm. In carrying out this analysis, von Stackelberg generally distinguishes between single production and joint production (von Stackelberg 1932: 10). By ‘joint production’ (in German: ‘verbundene Produktion’) von Stackelberg (1932: 5) denotes ‘the production of several kinds of products’ within one firm, irrespective of whether they are necessarily jointly produced or not. This very wide notion of joint production includes the case in which a firm operates several completely independent production processes each of which yields one single output (called ‘parallel production’). Another special case is when one of the outputs cannot be produced without jointly producing another one (called ‘linked production’, in German: ‘Kuppelproduktion’).

The technology of the firm is described in terms of its cost function, $K(x_1, \ldots, x_n)$, which denotes the minimal total costs of producing the amounts $x_1, \ldots, x_n$ of $n$ different outputs. The cost function is assumed to be regular, i.e. continuous and continuously differentiable, and monotonic (von Stackelberg 1932: 22, 54). Implicitly, decreasing returns to scale are assumed, such that a profit maximising production programme with positive and finite output exists. Throughout the analysis only variable costs are considered, such as the costs for labour force or raw materials. Fixed costs, as given by e.g. the use of durable capital goods, are neglected (von Stackelberg 1932: 9). As far as the firm’s market position is concerned, von Stackelberg (1932: 16–20) assumes that the inverse demand functions $P_i = P_i(x)$ for all outputs $i \ (i = 1, \ldots, n)$ are given. This assumption covers the two special cases that (i) the firm is a monopolist on the market for product $i$, or that (ii) the firm is a price-taker on that market, i.e. $P_i = \text{const.}$ That is, von Stackelberg provides a partial equilibrium analysis of the firm’s profit-maximising choice of the quantities to be produced of all of its products, given its cost function and given the inverse demand functions for all products.

Von Stackelberg conceptualises joint production by introducing two new theoretical tools: the notions of ‘length’ and ‘direction’ of the production vector $\mathbf{x} = (x_1, \ldots, x_n)$ (von Stackelberg 1932: 54–5). They are obtained by replacing the Cartesian coordinates $x_1, \ldots, x_n$ by polar coordinates. In the case of just two outputs, for which von Stackelberg defines these terms, $x_1$ and $x_2$ are replaced by $r$ and $\varphi$ with $r^2 = x_1^2 + x_2^2$ and $\tan \varphi = x_2/x_1$. A production vector $\mathbf{x}$ is then uniquely determined by its length $r$ and its direction $\varphi$.

Having introduced polar coordinates, von Stackelberg first studies the case in which the output proportion $x_2/x_1$ of the joint products is fixed. In this case the direction $\varphi$ of all different feasible production vectors is the same and constant, and they only differ in their length $r$. In this case the
determination of the optimal production programme is formally no different from the case of single production. The fact that more than one output is produced is irrelevant since a bundle of joint products, which are produced in fixed proportions, can be seen as an entity, constituting one fictitious product. Von Stackelberg calls this fictitious product a ‘packet’ (von Stackelberg 1932: 55). Writing the production vector as \( \mathbf{x} = r \mathbf{e}_\phi \) (von Stackelberg 1932: 56) where \( r \) is the length of the production vector and \( \mathbf{e}_\phi \) is the unit vector in direction \( \phi \), one can describe the production of this firm as resulting in a certain number \( r \) of packets. Since the output proportion \( x_2 / x_1 \) in the packet remains constant, i.e. the unit vector \( \mathbf{e}_\phi \) is constant, costs of production and sales, and thus the entire decision problem of the firm, depend only on the number \( r \) of packets produced. Von Stackelberg (1932: 57) summarises this result in what he calls ‘the fundamental proposition of joint production’:

\[ \text{[Proposition] (XXVI)} \]

With fixed proportions of the goods produced, all laws of single production remain valid for joint production, if one replaces the velocity of production of single production by length \([\ldots] \) of the production vector. (von Stackelberg 1932: 57)

He concludes that by this proposition the case of joint production with fixed output proportions is completely reduced to the case of single production.

In the case of joint production with flexible output proportions, firms not only can decide about the length \( r \), but also about the direction \( \phi \) of the production vector \( \mathbf{x} \). Von Stackelberg analyses the profit-maximising choice of the production vector as a choice over both \( r \) and \( \phi \). He employs a graphical method of analysis that is based on isocost curves and isorevenue curves in output space. In general, isocost curves may be convex or concave. Isorevenue curves may have a variety of shapes (von Stackelberg 1932: 75, Footnote 1) but in the special case that the firm is a price-taker on both output markets they are downward sloping parallel lines. Under the two assumptions that (i) isocost curves be strictly concave, i.e. average output bundles are less costly to produce than extreme output bundles, and (ii) the firm is a price-taker on the output markets, i.e. isorevenue curves be linear, von Stackelberg (1932: 62–3) derives a so-called ‘curve of the most favourable directions’ in output space. This curve is defined as the locus of all those production programmes \((x_1, x_2)\), which yield a given revenue at lowest possible total costs. It is obtained in the following way. If one draws the set of strictly concave isocost curves and the set of linear isorevenue curves in the same \( x_1 x_2 \)-diagram, then every isorevenue curve is intersected by infinitely many isocost curves. For every isorevenue curve there is an innermost isocost curve, which has exactly one point in common with that isorevenue
curve. The locus of all these points is the ‘curve of the most favourable
directions’.\textsuperscript{17}

In a two stage optimising procedure, firms would first derive the curve of
the most favourable directions in output space, which, under the two
assumptions specified above, always exists and uniquely determines the
direction of the production vector in terms of its length. By this first step
the direction of the most favourable production vector is eliminated from
the decision problem. In a second step, firms then choose the profit-
maximising output combination on this curve; that is, they choose the
optimal length of production given the curve of the most favourable direc-
tions. The problem of choosing the optimal length of production for a
given direction, however, has already been solved in the context of joint
production with fixed output proportions. He concludes

that by this construction the general case of joint production [i.e. joint production
with flexible output proportions] is reduced to the special case of joint [i.e. fixed]
output proportions, and from this case according to proposition XXVI to the case of
single production of one good.

(von Stackelberg 1932: 64–5)

Thus, the reduction of the analysis of the most general case of joint pro-
duction to the case of single production seems to be complete. However,
it should be noted that this general theory of the optimal production
decision of a joint production firm nevertheless requires two genuinely
new analytical tools which do not exist in the theory of single produc-
tion. The first tool is the ‘curve of the most favourable directions’, which
allows the analysis of the case of joint production with flexible output
proportions in terms of the theory of joint production with fixed output
proportions. The second tool is the notion of a ‘product packet’, which
allows the analysis of the choice of the optimal length under joint pro-
duction with fixed output proportions in terms of the theory of single
production.

Besides the graphical analysis, von Stackelberg (1932: 65–9) formally
derives necessary conditions for a profit maximum in the general case of joint
production in flexible output proportions. To this end, von Stackelberg
returns to a functional representation of costs and revenues in terms of
Cartesian coordinates, i.e. output quantities $x_1$ and $x_2$.\textsuperscript{18}

In the case of perfect competition, where output prices are constant and
exogenous to the firm’s decision, the marginal costs of producing each
product should equal its price (von Stackelberg 1932: 67). This result essen-
tially corresponds to Marshall’s result (cf. Section 2). In the case of monop-
oly where, in general, both output prices $P_1$ and $P_2$ depend on the amounts
produced of both outputs, the necessary conditions for a profit maximum
are (von Stackelberg 1932: 68):
\[
\frac{\partial K(X_1, X_2)}{\partial x_i} = P_i(X_1, X_2) + X_1 \frac{\partial P_1(X_1, X_2)}{\partial x_i} + X_2 \frac{\partial P_2(X_1, X_2)}{\partial x_i} (i = 1, 2).
\]

While \(\partial P_i / \partial x_i\) is always negative, the sign of \(\partial P_i / \partial x_j\) \((i \neq j)\) may be positive, zero or negative, depending on the character of the two outputs. If they are substitutes in consumption, \(\partial P_i / \partial x_j\) \((i \neq j)\) is negative and the marginal costs of the profit-maximizing output quantities are always smaller than the respective output prices. If, in contrast, the two outputs are complements in consumption, \(\partial P_i / \partial x_j\) \((i \neq j)\) is positive and the marginal costs of the profit-maximizing output quantities may be smaller or greater than the respective output prices. In the special case of monopoly, where the output price for each of the two joint products depends solely on the amount produced of that particular output, the necessary conditions for a profit maximum are:

\[
\frac{\partial K(X_1, X_2)}{\partial \lambda_i} = P_i(X_1) + X_i \frac{\partial P_i(X_1)}{\partial \lambda_i} (i = 1, 2).
\]

In this case, since \(\partial P_i / \partial x_i\) is always negative, it is immediately obvious that the marginal costs of the profit-maximising output quantities are smaller than the respective output prices.

4. Programmatic (mis)interpretation

The programmatic conclusion as to how to model production in economic theory, which von Stackelberg (1932: 75) draws from the study of the two cases of joint production with fixed and with flexible output proportions is that

[the theory of joint production [...] allows us to use the image of single production for all sectors of an economy. All statements about the economy, which are based on the assumption of single production, are not altered by the fact, that joint production exists as well: [...] they only need to be supplemented. This enables us greatly to simplify our analysis: we only need to deal explicitly with the problem of joint production where the direction of production itself becomes subject of the theory.

While this conclusion certainly summarises von Stackelberg’s results about joint production in a correct way, it is crucial to note that the first part, which refers to the case of joint production with fixed output proportions, is only true in combination with the second part, i.e. that the theory has to be supplemented in the general case of joint production with flexible output proportions. Without this supplement, the statement in the first part becomes misleading, if not erroneous.

Indeed, the second part of the conclusion has sometimes been ignored. One early example is the influential textbook on production theory by
Erich Schneider (1934), one of the cofounders of the Econometric Society. In this text he does not give any treatment at all to the issue of joint production, which is justified by a reference to von Stackelberg’s work (Schneider 1934: iii). Even Hans Möller, who was von Stackelberg’s assistant in Berlin, is a victim of the misinterpretation. In a review article on the scientific opus of von Stackelberg he judges that

[t]he decisive contribution of von Stackelberg to cost theory consists in showing that the theoretical statements about costs under single production can be applied as well under joint production.

(Möller 1949: 402; my translation)

Von Stackelberg himself contributed to the impression that the main result lies in the first part of his programmatic conclusion by downplaying the second part. Based on the first part of the above conclusion, von Stackelberg (1943, 1948, 1952) gives the issue of joint production a rather short treatment in his textbooks. For instance, the chapter on joint production in The Theory of the Market Economy (1952: Ch. 5 of Part II) is introduced by the following remark:

From the theory of costs in simple [that is: single] production we can derive a number of rules and build up in an abbreviated form the theory of joint production.

(von Stackelberg 1952: 69)

The whole exposition of joint production in this text is guided by von Stackelberg’s stressing the analogy between a theory of joint production (with fixed output proportions) and the theory of single production. Yet, at no point in the textbook does he make an explicit distinction between the cases of fixed and flexible output proportions. At the end of the section on costs in joint production von Stackelberg (1952: 71) draws the conclusion:

We observe therefore that the analogy between simple [that is: single] production and joint production is complete.

This abbreviated treatment of joint production in the textbook is, as it stands, misleading.

While von Stackelberg in his Kostentheorie (1932) applies a partial equilibrium analysis, he addresses the issue of general equilibrium in his other main work, Marktform und Gleichgewicht (1934), upon which his reputation is based.20 Von Stackelberg analyses equilibrium prices and quantities under different forms of competition on the market. However, this general equilibrium analysis is essentially built upon the assumption of single production:

In the following we shall predominantly be concerned with the case of single production and exclude joint production. This is […] justified because it can be shown
that the fundamental problems of this kind of production can be reduced to single production.

(von Stackelberg 1934: 5; my translation)

This conclusion, however, cannot be justified for two reasons. First, as pointed out above, the final conclusion of the *Kostentheorie* comprises two parts, to the first of which reference is made in order to justify this simplification, and the second of which, clearly states that the first part does not contain the whole truth. And second, the results in the *Kostentheorie* are derived in a partial equilibrium analysis only. Combining several outputs jointly produced to form a composite good may be appropriate to determine the optimal production programme for one individual firm and all prices given. But it is certainly inappropriate for the discussion of general equilibrium when quantities *and* prices for the joint products are to be determined simultaneously from the behaviour of different firms and consumers (Buchanan 1966: 405; Malinvaud 1985: 62, footnote).  

5. Assessment of von Stackelberg’s contribution

Von Stackelberg, in his *Kostentheorie* (1932), provided a contribution to the theory of joint production which is very modern in its style and concepts. It constitutes the first comprehensive and formally rigorous treatment of the decision problem of a multi-output firm which covers the different cases of perfect competition and monopoly on the market for the joint products. Subsequent textbooks and monographs on cost and production theory (e.g. Carlson 1939, Frisch 1965, Krelle 1969, Schneider 1972[1949]) largely follow von Stackelberg in their treatment of joint production and explicitly acknowledge his pioneering contribution. Yet, von Stackelberg’s (1932) analysis of joint production – just like Marshall’s (1925), and in contrast to Fanno’s (1999[1914]) – is restricted in that it remains strictly within a partial analysis framework. His attempt to carry over his insights to a general equilibrium analysis (von Stackelberg 1934) is untenable. Inasmuch, von Stackelberg probably overestimated the generality of his results.

While some of von Stackelberg’s results were obtained earlier by others (e.g. by Marshall, Fisher, Edgeworth and Fanno), his most original and fruitful contribution may be seen in the introduction of polar coordinates. This allows the study of joint production in terms of the ‘length’ and the ‘direction’ of production. With the aid of these two concepts it is possible to separate the discussion of the output bundle’s composition from the discussion of the scale of production. Niehans (1992), while acknowledging the originality of von Stackelberg’s approach, nevertheless is sceptical as to
the usefulness of the two concepts of length and direction of production. In his opinion, it only makes sense to separate the choice of structure from that of scale for homogeneous production functions. For more general functions, according to Niehans (1992: 193), the transformation of the original function in terms of several inputs and/or outputs into a function in terms of structure and scale offers no analytical advantage.

This judgement is in striking contrast to the actual development in the modern industrial organization literature dealing with multi-output firms and industries (e.g. Baumol 1977, Panzar and Willig 1977a, b, 1981 or Willig 1979). It is typical for large parts of this literature that the behaviour of firms is analysed in terms of their costs without explicitly considering technical descriptions of production, such as production functions. In particular, the analysis is not confined to the narrow case of homogeneous production functions. However, the analysis in terms of structure and scale – although not explicitly carried out in polar coordinates – has been shown to yield new insights even in the very general case. Indeed, the often complicated interrelationship between structure and scale of output for the costs of the firm can be most clearly analysed by analytically separating the issues of economies of scale and economies of scope, the first concept referring to the scale of output and the second one to the structure or composition of the output. This procedure is perfectly in line with the analytical distinction between the two concepts of length and direction of the production vector as introduced by von Stackelberg. The summary that Bailey and Friedlaender (1982) give in their review article on the analysis of costs of multi-output firms or industries, based on the concepts of economies of scale and economies of scope, is worthy of being quoted here, since it is in remarkable agreement with the conclusion proposed by von Stackelberg half a century earlier:

To conclude, the cost function of a multi-product firm will be sensitive to the composition of output as well as to the scale of output. Thus, as the firm changes its level of output and its product mix, costs will typically change in a variety of fashions [. . .] Thus, the global cost surface may often exhibit properties that cannot be represented by the traditional concepts designed to indicate the structure of costs in a single-product setting.

(Bailey and Friedlaender 1982: 1032–33)

6. Conclusion

While the modern research on multi-product firms and industries has confirmed that Heinrich von Stackelberg’s analysis of the issue was both relevant and correct, it is hard to understand why von Stackelberg, who had clearly recognised a general theory of joint production covering both fixed
and flexible proportions to go well beyond any theory of single production in requiring additional formal elements, always in his later works restricted himself to the case of single production. One reason might be that he had very well noticed that a general theory of joint production is rather complicated. Von Stackelberg’s work on joint production, his attitude toward the issue, and the programmatic consequences of his analysis, cannot better be captured than by his own words with which he summarises the topic in his textbook *Grundzüge der theoretischen Volkswirtschaftslehre* (von Stackelberg 1943: 26; my translation):

The theory of joint production is an important chapter of economics. However, it is rather messy. For that reason, in the following only single production is treated in detail.

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**Notes**

1 Möller (1949: 396) reports that von Stackelberg’s *Kostentheorie* served, together with other texts, as the basis for degree exams at the London School of Economics as early as 1933/1934.

2 My translation.

3 An exception can be found in the strand of literature which goes back to Sraffa (1960) and advocates a so-called ‘neo-Ricardian’ approach to the theory of value and distribution (e.g. Schefold 1989, Kurz and Salvadori 1995, Pasinetti 1980, Salvadori and Steedman 1990).

4 Among the pioneering contributions to the literature on multi-product industries and market structure are Baumol (1977), Beckenstein (1975), Panzar and Willig (1977a, 1977b, 1981), Sharkey and Telser (1978) and Willig (1979). The theoretical side has been most fully expounded by Baumol *et al.* (1988). For an overview see Bailey and Friedlaender (1982).

5 For instance, the entry on von Stackelberg in *The New Palgrave: A Dictionary of Economics* (Krelle 1987) does not even mention von Stackelberg’s work on this issue. Only his work on price theory under imperfect competition, *Marktform und Gleichgewicht* (von Stackelberg 1934), is well known as a pioneering contribution to the literature on imperfect competition.

6 Von Stackelberg’s results, to be sure, are only one reason among many for this development. For a more comprehensive discussion of why the issue of joint production has
almost completely been omitted from the mainstream of modern economic theory, see Baumgürtner (2000: Ch. 8).

7 A second, to a large extent revised, edition appeared in 1948 under the title Grundlagen der Theoretischen Volkswirtschaftslehre and was subsequently translated into English as The Theory of the Market Economy (1952).

8 This and all subsequent translations from von Stackelberg (1932) are mine.

9 In his textbooks (1943: 25–6, 1948: 31–2, 1952: 29) von Stackelberg presents a systematic and elaborate taxonomy of single and joint production, which is essentially that developed earlier by Edgeworth (1925a[1911]: 84ff., 1925b[1915]: 485ff.).

10 While von Stackelberg distinguishes between cost functions which correspond to the three cases of increasing, constant and decreasing returns to scale in the introduction to the chapter on single production (von Stackelberg 1932: 20–3), the subsequent treatment in the Kostentheorie, and in particular the chapter on joint production, implicitly assumes decreasing returns to scale.

11 Von Stackelberg restricts himself to the case of just two outputs. While justifying this simplification by the ease of being able graphically to represent the analysis, he nevertheless explicitly notes that already with two joint products all problems and methods become apparent and this case can easily be generalised to \( n \) outputs (von Stackelberg 1932: 54).

12 This result had already been formally established earlier by Marco Fanno (1999[1914]). Von Stackelberg (1932: 53) acknowledges that Fanno (1999[1914]) provides a general theory of joint supply. However, he does not relate his own treatment in detail to Fanno’s.

13 The idea that several products which come in fixed proportions can be treated as an entity had already been evoked by Irving Fisher (1892: 65–6) for the case of joint demand. It should be noted that while such a procedure seems to be plausible from a purely formal point of view, the concept of a product packet turns out to be ‘difficult to define either conceptually or operationally’ (Bailey and Friedlaender 1982: 1029) as soon as applied to empirical examples of joint production.

14 Throughout his Kostentheorie von Stackelberg refers to output as a flow quantity by ‘velocity of production’, which denotes output per time.

15 Fisher (1892: 75–6) carries out a similar analysis, not in terms of the costs but in terms of the ‘disutility’ of jointly producing two outputs. Figure 26 in Fisher (1892: 76) therefore already contains some elements of von Stackelberg’s (1932: 64, Figure 13) graphical analysis.

16 In modern language one would say that economies of scope prevail (Panzar and Willig 1981; cf. the discussion in Section 5 below).

17 Von Stackelberg notes (1932: 64) that if isorevenue curves are linear and isocost curves are strictly convex – which means that average output bundles are more costly to produce than extreme output bundles – then the curve of the most favourable directions cannot be in the interior of the output space. Rather, the intersections between isorevenue curves and the associated innermost isocost curves occur on the axes. This means that

**[Proposition] (XXXII)** . . . under perfect competition joint production can never take place, but either good 1 or good 2 are exclusively produced.

(von Stackelberg 1932: 64)

The result, of course, hinges upon the possibility that the two goods can be produced independently. This possibility is contained in von Stackelberg’s very wide notion of ‘joint production’.
18 Von Stackelberg (1932: 65) justifies the use of polar coordinates by the statement that ‘this was the only way to demonstrate that the ideas formulated for single production hold as well for joint production’. This task being accomplished there is no more need for using polar coordinates.

19 Note that there is an obvious mistake in the corresponding formula of von Stackelberg (1932: 68).

20 Von Stackelberg had early on recognised the necessity for carrying out general equilibrium analyses (Möller 1949: 409). Immediately after completing the Grundlagen einer reinen Kostentheorie, von Stackelberg studied the general equilibrium price theory of Gustav Cassel (von Stackelberg 1933), who had popularised Walrasian general equilibrium in German-speaking countries. In his essay on problems of imperfect competition (von Stackelberg 1938b) he criticised the Anglo-Saxon approach to price theory, which goes back to Alfred Marshall and preferred partial over total analysis, and concluded that partial analysis had to be based on and controlled by a general equilibrium analysis.

21 To point to just one problem which joint production might bring about in a general equilibrium framework, under joint production the gross substitution property is less likely to hold. As a consequence, statements about the uniqueness and stability of general equilibrium prices and allocation are weakened (Müller-Fürstenberger 1995: 318–28).

22 Frisch (1965) does not make any reference to von Stackelberg. However, he presumably knows von Stackelberg’s results from the texts of Carlson (1939) and Schneider (1934, 1972[1949]) from which he quotes on several occasions.

23 Economies of scale can be captured in a multi-output setting by considering the two related concepts of ray economies of scale (Baumol 1977, Panzar and Willig 1977b) and product-specific economies of scale (Panzar and Willig 1977a). Ray economies of scale are a straightforward extension of the concept of single-product scale economies and indicate the behaviour of costs as the production levels of a given bundle of outputs change proportionately; that is, the composition of output is assumed to remain fixed while its scale is permitted to vary. Product-specific economies of scale measure how costs change as the output of one commodity changes while the output of all other commodities remains constant. While none of these measures of scale economies properly reflects the effect of the output mix upon costs, this is achieved by two more global measures, the concepts of economies of scope (Panzar and Willig 1981) and of transray convexity of the cost function (Baumol 1977). One speaks of economies of scope if the cost of producing several outputs jointly is less than the cost of separate production. An interesting result (Willig 1979) is that sufficiently strong scope economies may confer ray economies of scale among several outputs even if there are product-specific constant returns or diseconomies of scale. In this case one speaks of transray convexity of the cost function. That is, as a firm changes the composition of output while holding fixed the level of some aggregate measure of output, costs will be lower for diverse rather than specialised output mixes. For a more technical introduction into these concepts and results see Bailey and Friedlaender (1982: 1028–33).

References


Abstract

In the 1970s, considerable interest arose into the study of multi-output firms and industries. However, this literature did not seem to be aware of the contribution that von Stackelberg made to the issue almost half a century earlier. This paper outlines von Stackelberg’s contribution to the theory of costs under joint production. It critically assesses the place of his contribution in the modern history of the theory of joint production and it suggests an answer to the question of why von Stackelberg’s theory of joint production fell into oblivion and even contributed to the abandonment of the issue for decades.

Keywords

cost theory, Heinrich von Stackelberg, joint production, theory of the firm